NON-DEGENERATE STATISTICAL APPROACH TO THE BINDING ENERGY OF NUCLEI AND OF Λ-HYPERNUCLEI (II)

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ABSTRACT. The binding energy per nucleon of a hypothetical nucleus, without the coulomb and asymmetry energy corrections, is considered to be the mean value of neutron-proton interaction energy between neighbours The numbers of neutrons and protons in the hypothetical nucleus are considered to be equal (to be modified when coulomb and asymmetry energies are taken into account) and thus the binding energy per nucleon, $E^0{}_n(A)$, is considered to be approximately constant for all the nucleons in a nucleus In momentum space those nucleons are considered to be contained in different momentum shells, according to the nucleon numbers in the nucleus This gives rise to the total number of combination states in different nuclei, which along with the modified non-degenerate Fermi distribution function determines the probable number of occupied states To correspond to the required values of $E^0{}_n(A)$ (Dutta, et al, 1965) the parameters in the relation are determined The statistical relation, read as a relation to determine binding energy of neutrons, helps us in finding the relation for the binding energy of Λ-particles, in accordance with the one required empirically.

The binding energies of nuclei, in ground state, consists of a major portion dependent on the total nucleon number "A" and the associated correction terms which are dependent on Z for coulomb repulsion and on (N Z) for asymmetry. The Bethe-Weizsacker relation or its proposed modification (Dutta *et al* 1965), put the binding energies of such nuclei primarily dependent on these three characteristics of nuclear composition It should be a realisable and comparatively simpler procedure to find an expression for the binding energy of the total nucleon number in a nucleus, without the associated corrections for coulomb repulsion and asymmetry and then to ascertain the corrections.

It may be observed from the empirically fitted Bethe-Weizsacker relation (Green, 1958) or from the modified relation, that the binding energy per nucleon in a nucleus, in ground state, before the aforesaid corrections are made, increases with nucleon numbers. This is presumably on account of the decrease in the proportion of surface nucleons. It would be expected that both the kinetic and the potential energy per nucleon, in ground state, contribute towards this increase in energy per nucleon.

This binding energy per nucleon, might be caused by the interaction between neutrons and protons, when they are neighbours and react by exchange forces. This is in accordance with the view of Bethe and Bacher (1936), as stated on page 150 and 95 of their communication, that "any given nuclear particle interacts essentially with two (neighbouring) particles of the other kind" and that "the nuclear forces have the character of exchange forces between neutrons and protons". On such a basis, when we do not take into account the asymmetry or coulomb energy terms, we have the hypothetical case of nuclei composed of equal or nearly equal numbers of interacting protons and neutrons, such that half the binding energy of a neutron proton pair or rather one fourth the binding energy between two protons and two neutrons, which are neighbours, is regarded as the binding energy per nucleon. The corrections bring down the binding energy of the hypothetical nucleus to the actual value obtaining there

The energy per nucleon of the hypothetical nucleus, thus considered, is expected to be fairly constant for all the nucleons both in respect of their kinetic and potential energy constituents. For nuclei with larger nucleon numbers both the constituents increase, in accordance with our previous observation Thus, for a particular nucleus, without the corrections, we may take the potential energy per nucleon $V_n^0(A)$, as approximately constant, giving us immediately the total potential energy of the nucleus This eliminates the usual process of potential energy calculation by relations of the form :,

$$V = \sum_{i=1}^{Z} \sum_{k=1}^{N} \psi_i^*(x_1) \psi_i(x_2) \varphi_k^*(x_2) \varphi_k(r_1) J(r_{12}) d\tau_1 d\tau_2 \tag{B}$$

as suggestend by Bethe and Bacher (1936), where $\psi_i(x_i)$ and $\varphi_k(x_k)$ refer to the solutions of one particle wave equations for the protons and neutrons and $J(r_{12})$ is the interaction potential The kinetic energy $T_n^0(A)$ per nucleon in the ground state, similarly, is considered to be fairly constant for all the nucleons in a nucleus It should, however, have a range to accomodate the different energy states of the nucleons, which are exclusive. It would naturally require larger range of energy for nuclei with larger nucleon number Other possible dispositions of nucleons for any particular nucleus would have less binding energy and the associated kinetic energy. The total energy per nucleon $E_n^0(A)$, in a nucleus in the ground state may, however, be considered to be determined by the sum,

$$E_n^0(A) = T^0{}_n(A) + V^0{}_n(A). \qquad \dots \quad (1)$$

We may, thus, state that if the nucleons are distributed in momentum space, the corresponding values for all the nucleons in the nucleus, in ground state, would lie in a momentum shell $4\pi(p^0{}_n)^2dp$, rather than be distributed to fill up the momentum space $\frac{4}{3}\pi p^3$, with p varying from 0 to p_{max}, which is the usually adopted procedure (Fermi 1950)

The number of states for such neutrons and protons in the nucleus, when spin degeneracy is taken into account, would be determined by the expressions,

$$\frac{8\pi\Omega}{h^3}\,(p^2{}_N\,dp_N)^0 \text{ and } \frac{8\pi\Omega}{h^3}\,(p^2{}_P\,dp_P)^0,$$

where Ω is the enclosing three dimensional space. The neutron-proton combination states in the shell, where each of the neutron states could combine with each of the proton states, would be given by the product as

$$C_J = \left(\frac{8\pi\Omega}{h^3}\right)^2 (p^2{}_N\,dp_N)^0\,(p^2{}_P\,dp_P)^0 \qquad \ldots \quad (2)$$

Such a combination of neutron and proton states is also envisaged in the expression for the potential energy V, in eqn (B) above, used by Bethe and Bacher

We consider Ω as well as $(p^0{}_n)^2$, which increases with A, to be primarily proportional to A In the case of small nuclei with less than about fourteen nucleons, however, the nuclear volume in some nuclei or the binding energy in others, is known to decrease much more slowly than required by the proportionality with A (Hofstadter, 1956) Such irregularity in the range of small nuclei may be taken into account by a term of the form $f(A) = \{1 + S(A)\}$, as an additional factor in the converted equation (2) for cell numbers. It is considered that the structure dependent constituent $S(A)$ establishes itself comparatively slowly, at the dynamically equilibrium condition of the nucleus, in ground state It gives us the total number of possible combination cells for a nucleus, given by nucleon number as

$$C(A) \propto A^2 f(A) \int_0^{P^0{}_N} \int_0^{P^0{}_P} (p^2{}_N\,dp_N)(p^2{}_P\,dp_P)$$

$$\propto A^5 f(A) \qquad \ldots \quad (3)$$

Let us now consider the distribution of N combinations of neutron and protons, with the total energy E. The combinations may be placed in different energy shells, corresponding to the nuclei of different 'A' values. The number of cells in a particular shell is $C(A)$, with $N(A)$ nucleons of one type, in each combination that is associated with the energy $2E^0{}_n\,(A)$, for the shell. The conditions

$$\Sigma\, N(A) = N \;;\; \Sigma\, N(A)\cdot 2E^0{}_n\,(A) = E$$

and the distribution number satisfying Fermi or Bose distribution,

gives the usual form of most probable distribution function,

$$N(A) = C(A)\left\{\exp\left(\frac{E^0{}_n\,(A) - E_F}{kT}\right) \pm 1\right\}^{-1}. \qquad \ldots \quad (4)$$

We consider the problem to be nondegenerate for nuclei, with $A \geqslant 4$, on account of the large nucleonic energy and mass This would give us the simplified nondegenerate statistical relation connecting the number of occupied states and the ground state energy of the nucleons in different nuclei as,

$$N(A) \propto A^5 f(A) \exp\left(\frac{-E^0{}_n(A)+E_F}{KT}\right) \tag{5}$$

We have considered here that the number of occupied single states is $A/2$, with the energy $E^0{}_n(A)$, associated with the occupied combination states $N(A)$ in a nucleus. If we restrict ourselves to these single states, we have the more explicit form of equation (5) as,

$$A/2 = kA^5 f(A) \exp[-E^0{}_n(A)/KT] \quad \dots \ (6)$$

$$1 = 2k \cdot A^4 f(A) \exp[-E^0{}_n(A)/KT], \quad \dots \ (6a)$$

where $\exp E_F/KT$ has been incorporated in the constant of proportionality.

The relation (6) implies that half the occupied single state nucleonic cells equals the probability of occupation of cells with the energy of a single neutron or proton, out of the total cell number Further since 1/2 unit of occupied single state neutron and proton cells measures 1 unit of occupied neutron or proton cell, we may put the equation (6) also in the form

$$1.\ (\text{occupied neutron cell}) = kA^4 f(A) \exp[-E^0{}_n(A)/KT] \tag{6b}$$

The equation (6a) is very suitable to determine the binding energy $E^0{}_n(A)$ of a nucleon in ground state, for different nucleon numbers A. The parameters k, KT and $f(A)$ have been determined such that the values of $E^0{}_n(A)$, determined from the relation (6a), more or less agree with the values obtained in a previous work (Dutta *et al* 1965) where the values of $E_n{}^0(A)$ along with the expected amounts of the coulomb and asymmetry energy contributions, give us the binding energies of nuclei, in close approximation to the observed magnitudes. We, thus, obtain *eq.* (6a) in explicit form as

$$1 = 2\times 2.327\times 10^4\{1+S(A)\}A^4 \exp[-E^0{}_n(A)/0.40] \quad \dots \ (6c)$$

with $$S(A) = 4.2 \exp[-3.27\times 10^{-2}A^2]$$

The required (KT) value varies from 6 to 3 per cent of nucleonic energy.

It may be stated that too large coulomb repulsion energy for larger nuclei, with equal proton and neutron numbers, and a consequent too large internucleonic separation for nucleonic interaction to be effective, introduces the asymmetric nucleons which are not as strongly bound This brings in the associated corrections. It may also be noted that such non-degenerate statistical approach to the binding energy of the nuclei automatically incorporates the Bethe-Weiszacker relation, from which we started. The numbers of protons and neutrons in a nucleus and the magnitudes of the asymmetry and coulomb energy corrections are to be determined by considering proper functional forms of the two corrections in terms of A and Z and finding the particular value of Z that makes the sum of the corrections a minimum

The statistical relations discussed so far must necessarily be concerned with the determination of the number of particles occupying different energy states, in the equilibrium condition One can not extend these statistical relations, as such, to nonequilibrium condition We may, however, consider these statistical relations in the reverse way, to determine the energy per nucleon in the ground state in terms of a function of the nucleon number A. Such a consideration may then be extended to include the non-equilibrium states also It would be seen from eq. (7) below, that the ground state binding energy of a neutron is determinable in units of $E^0_N(A)$, by a measure expressed in terms of A An appropriate change in the functional form of the measure, would give us the binding energy in the nonequilibrium states also, in units of ground state energy $E^0_N(A)$ Thus, we may rewrite eq (6b) in the equivalent form

$$E^0_N(A) \cdot 1 = E^0_N(A)[2.327 \times 10^4 A^4\{1 + S(A)\} \exp(-E^0_N(A)/0.40] \quad . \qquad (7)$$

where the values of k and KT ascertained in eq. (6c) have been incorporated. The functional form associated with $E^0_N(A)$, on the right hand side, is the measure of ground state energy in units of $E^0_N(A)$. The binding energy per neutron $E^0_N(A)$, in non-equilibrium state, would be determined by the correspondingly changed functional form in the expression for the measure. In the case of the neutral Λ-particles we consider that the comparativly slow structure dependent constituent $S(A)$ drops out and that the parameter (KT) is suitably modified This would give us the binding energy of the Λ-particles in different associations, as

$$\Lambda_{B_E} = E^0_N(A)[2.327 \times 10^4 A^1 \exp[-E^0_N(A)/(kT)_\Lambda]$$

$$= 2.327 \times 10^4 E^0(A) \cdot A^3 \exp[-E^0_N(A)/0.42], \qquad \ldots \quad (7a)$$

where $E^0(A)$ is the uncorrected total binding energy of the nucleus, $E^0_N(A) \times A$ and $(KT)_\Lambda$ as 0.42Mev, has been ascertained by trial. A similar expression on empirical approach, has been found in the previous note (Dutta *et al* 1, 1966), to be suitable for the determination of the binding energies of the Λ-hyperons.

The calculated values of $E^0_N(A)$ and Λ_{B_E} by equations (6c) and (7a), along with the previously obtained values (Dutta *et al* 1965) of $E^0_N(A)$ and the experimental values of Λ_{B_E} (Levi Setti 1965), are tabulated below :

TABLE I

A	$E^0_n(A)$ eq (6c)	$E^0_n(A)$ (Previous)	ΛB_t eq.(7a)	ΛB_c (exp)
4	7.02	—	2 3	2.3
5	7 30	—	3 0	3.1
7	7 66	7.74	5.1	4 9— 5 5
8	7 79	7 80	6.5	6.4— 6 6
10	8 04	7.83	9 0	8 5—10 0
13	8 41	8 10	11.3	10.6
14	8 52	8.20	11 8	12 0—13.0
19	9 01	8 75	13.2	—
31	9 80	9 76	15 8	—
55	10 71	10 80	19 2	~20
111	11 83	11 92	24 3	~25
222	12 95	12.92	30 2	~30
250	13.14	13 03	31.4	~30

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